3.1 Types of Sets and Set Notation

YOU WILL NEED

- compass

EXPLORE...

- What categories can you use to organize your clothes?

set
A collection of distinguishable objects; for example, the set of whole numbers is $W = \{0, 1, 2, 3, \ldots\}$.

element
An object in a set; for example, 3 is an element of D, the set of digits.

universal set
A set of all the elements under consideration for a particular context (also called the sample space); for example, the universal set of digits is $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

subset
A set whose elements all belong to another set; for example, the set of odd digits, $O = \{1, 3, 5, 7, 9\}$, is a subset of D, the set of digits. In set notation, this relationship is written as:
$O \subset D$

complement
All the elements of a universal set that do not belong to a subset of it; for example, $O' = \{0, 2, 4, 6, 8\}$ is the complement of $O = \{1, 3, 5, 7, 9\}$, a subset of the universal set of digits, D. The complement is denoted with a prime sign, O'.

GOAL

Understand sets and set notation.

INVESTIGATE the Math

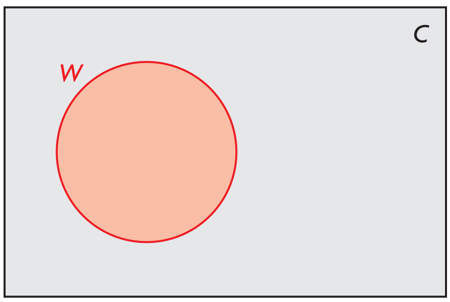
Jasmine is studying the provinces and territories of Canada. She has decided to categorize the provinces and territories using **sets**.

? How can Jasmine use sets to categorize Canada's regions?

A. List the **elements** of the **universal set** of Canadian provinces and territories, C.

B. One **subset** of C is the set of Western provinces and territories, W. Write W in set notation.

C. The Venn diagram to the right represents the universal set, C. The circle in the Venn diagram represents the subset W.



The **complement** of W is the set W'.

i) Describe what W' contains.
ii) Write W' in set notation.
iii) Explain what W' represents in the Venn diagram.

D. Jasmine wrote the set of Eastern provinces as follows:

$E = \{NL, PE, NS, NB, QC, ON\}$

Is E equal to W'? Explain.

E. List T, the set of territories in Canada. Is T a subset of C? Is it a subset of W, or a subset of W'? Explain using your Venn diagram.

F. Explain why you can represent the set of Canadian provinces south of Mexico by the **empty set**.

G. Consider sets C, W, W', and T. List a pair of **disjoint** sets. Is there more than one pair of disjoint sets?

H. Complete your Venn diagram by listing the elements of each subset in the appropriate circle.

empty set

A set with no elements; for example, the set of odd numbers divisible by 2 is the empty set.

The empty set is denoted by { } or ∅.

disjoint

Two or more sets having no elements in common; for example, the set of even numbers and the set of odd numbers are disjoint.

Reflecting

I. Why might you use a Venn diagram instead of a map to categorize the regions of Canada? Explain with an example.

J. **i)** What other sets could you use to sort the provinces and territories?
ii) Which of these new sets are subsets of the sets you used earlier?
iii) Which of these new sets are disjoint?

Communication | *Notation*

The following is a summary of notation introduced so far.

Sets are defined using brackets. For example, to define the universal set of the numbers 1, 2, and 3, list its elements:

$U = \{1, 2, 3\}$

To define the set A that has the numbers 1 and 2 as elements:

$A = \{1, 2\}$

All elements of A are also elements of U, so A is a subset of U:

$A \subset U$

The set A', the complement of A, can be defined as:

$A' = \{3\}$

To define the set B, a subset of U that contains the number 4:

$B = \{\ \}$ or $B = \varnothing$

$B \subset U$

APPLY the Math

EXAMPLE 1 Sorting numbers using set notation and a Venn diagram

a) Indicate the multiples of 5 and 10, from 1 to 500, using set notation. List any subsets.

b) Represent the sets and subsets in a Venn diagram.

Ramona's Solution

a) $S = \{1, 2, 3, ..., 498, 499, 500\}$

I defined S as the universal set of all natural numbers from 1 to 500.

$S = \{x \mid 1 \leq x \leq 500, x \in \mathrm{N}\}$

There were too many numbers to list, so I wrote an expression for the set. $1 \leq x \leq 500$ means that x can be any number from 1 to 500.

$F = \{5, 10, 15, ..., 490, 495, 500\}$
$F = \{f \mid f = 5x, 1 \leq x \leq 100, x \in \mathrm{N}\}$

I defined F as the set of multiples of 5 from 1 to 500. Since the greatest element of F is 500, I wrote the set as a multiple of 5 using values of x from 1 to 100.

$F \subset S$

F is a subset of S since all the elements of F are also elements of S.

$T = \{10, 20, 30, ..., 480, 490, 500\}$
$T = \{t \mid t = 10x, 1 \leq x \leq 50, x \in \mathrm{N}\}$

I defined T as the set of all multiples of 10 from 1 to 500.

Since the greatest element of T is 500, I wrote it as a multiple of 10 using values of x from 1 to 50.

$T \subset F \subset S$

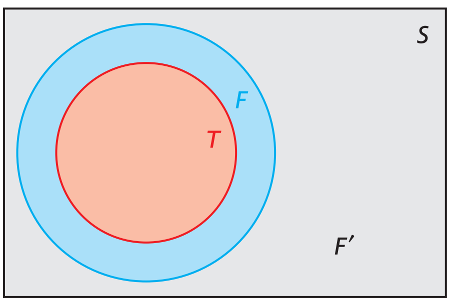
T is a subset of both F and S.

$F' = \{\text{non-multiples of 5 from 1 to 500}\}$

The complement of F, F', contains all the numbers that are not multiples of 5.

There are too many numbers to list, and I can't write "non-multiples of 5" using an algebraic expression, so I wrote a description.

b)



I showed the relationships among the sets in a Venn diagram.

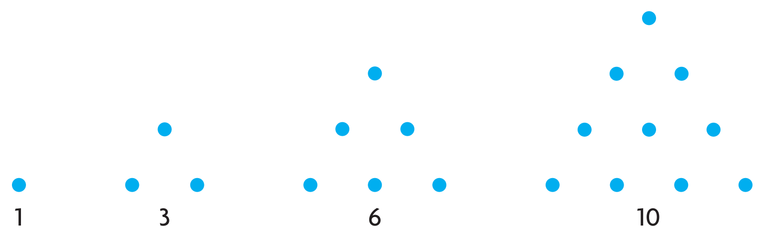
Since $T \subset F$, the circle that represents the subset T is inside the circle that represents the subset F.

Your Turn

Indicate the multiples of 4 and 12, from 1 to 240 inclusive, using set notation. List any subsets, and show the relationships among the sets and subsets in a Venn diagram.

EXAMPLE 2 Determining the number of elements in sets

A triangular number, such as 1, 3, 6, or 10, can be represented as a triangular array.



a) Determine a pattern you can use to determine any triangular number.
b) Determine how many natural numbers from 1 to 100 are
  i) even and triangular,
  ii) odd and triangular, and
  iii) not triangular.
c) How many numbers are triangular?

Simon's Solution

a) I examined the pattern of dots for the triangular numbers 1, 3, 6, and 10 to find a pattern.

1

The first triangular number is 1.

3 $\quad 1 + 2$

The second triangular number is the sum of the first two natural numbers.

6 $\quad 1 + 2 + 3$

The third triangular number is the sum of the first three natural numbers.

10 $\quad 1 + 2 + 3 + 4$

The fourth triangular number is the sum of the first four natural numbers.

The nth triangular number is the sum of the first n natural numbers.

Each triangular number is a sum of consecutive natural numbers, beginning at 1.

b) i) I used a spreadsheet to generate more triangular numbers.

	A	B
1	Natural Numbers	Triangle Numbers
2	1	1
3	2	3
4	3	**6**
5	4	**10**
6	5	15
7	6	21
8	7	**28**
9	8	**36**
10	9	45
11	10	55
12	11	**66**
13	12	**78**
14	13	91
15	14	105

I entered the natural numbers in column A.

I entered the first triangular number, 1, in cell B2.

To define the second triangular number, I entered the formula = B2 + A3 in cell B3.

I dragged the formula down until I reached a triangular number greater than 100.

I put all the even triangular numbers in boldface.

Communication *Notation*

The phrase "from 1 to 5" means "from 1 to 5 inclusive."

In set notation, the number of elements of the set X is written as $n(X)$.

For example, if the set X is defined as the set of numbers from 1 to 5:

$X = \{1, 2, 3, 4, 5\}$
$n(X) = 5$

$U = \{$natural numbers from 1 to 100$\}$
$T = \{$triangular numbers from 1 to 100$\}$
$E = \{$even triangular numbers from 1 to 100$\}$

I defined the universal set, U, and subsets T and E.

$T = \{1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91\}$
$E = \{6, 10, 28, 36, 66, 78\}$

I listed the elements in T and E.

$n(T) = 13$
$n(E) = 6$
There are 13 triangular numbers from 1 to 100, and 6 of these numbers are even.

I determined the number of elements in each **finite set** by counting.

finite set
A set with a countable number of elements; for example, the set of even numbers less than 10, $E = \{2, 4, 6, 8\}$, is finite.

ii) $O = \{$odd triangular numbers from 1 to 100$\}$
$n(O) = n(T) - n(E)$
$n(O) = 13 - 6$
$n(O) = 7$

There are 7 odd triangular numbers from 1 to 100.

I defined the set of odd triangular numbers, O.

Since triangular numbers are either even or odd, I subtracted the number of even triangular numbers from the total number of triangular numbers.

iii) $n(U) = 100$
$n(T') = n(U) - n(T)$
$n(T') = 100 - 13$
$n(T') = 87$

There are 87 numbers from 1 to 100 that are not triangular.

The universal set contains 100 elements. These elements are either triangular numbers or they are not.

I subtracted to determine the number of elements in subset T', the set of non-triangular numbers.

c) There is an infinite number of natural numbers, so there must be an infinite number of triangular numbers. The set of triangular numbers is an **infinite set**.

The pattern for triangular numbers continues forever. I can start to count the triangular numbers, but it is impossible to count all of them.

infinite set

A set with an infinite number of elements; for example, the set of natural numbers, $N = \{1, 2, 3, \ldots\}$, is infinite.

Your Turn

Explain why Simon defined O as a separate subset, rather than using E'.

EXAMPLE 3 Describing the relationships between sets

Alden and Connie rescue homeless animals and advertise in the local newspaper to find homes for the animals. They are setting up a web page to help them advertise the animals that are available. They currently have dogs, cats, rabbits, ferrets, parrots, lovebirds, macaws, iguanas, and snakes.

a) Design a way to organize the animals on the web page. Represent your organization using a Venn diagram.
b) Name any disjoint sets.
c) Show which sets are subsets of one another using set notation.
d) Alden said that the set of fur-bearing animals could form one subset. Name another set of animals that is equal to this subset.

Connie's Solution

a) I defined the universal set, A.

A = {all the animals that are available}
W = {warm-blooded animals}
C = {cold-blooded animals}

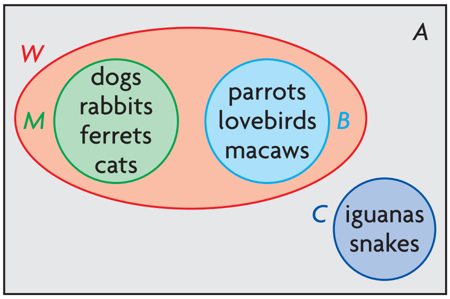
The animals are either warm-blooded or cold-blooded. I defined the subsets W and C for these two types of animals.

I listed the elements in each subset.

W = {dogs, cats, rabbits, ferrets, parrots, lovebirds, macaws}
C = {iguanas, snakes}

I decided to organize the subset of warm-blooded animals into two further subsets: mammals, M, and birds, B,

M = {dogs, cats, rabbits, ferrets}
B = {parrots, lovebirds, macaws}



I drew circles to represent subsets M (mammals), B (birds), and C (cold-blooded animals). Subsets M and B form the set W (warm-blooded animals), so I drew an oval around these two circles.

Then I drew a rectangle around all the shapes to represent the universal set, A (all available animals).

b) The disjoint sets are W and C, M and C, B and C, and M and B.

I looked for shapes that do not overlap. I knew that these sets do not contain common elements.

c) $M \subset W$, $B \subset W$
$M \subset A$, $B \subset A$, $C \subset A$, $W \subset A$

Sets M and B are inside W, so they are subsets of W. All the sets are subsets of the universal set, A.

d) $F = \{\text{fur-bearing animals}\}$
$F = \{\text{cats, dogs, ferrets, rabbits}\}$
$M = F$
Set M is equal to set F, the set of fur-bearing animals.

I defined set F. Then I listed the elements in F.

Set M contains the same elements as F, so these sets are equal.

It does not matter that the animals in M are listed in a different order than the animals in F.

Your Turn

How else might you categorize the animals into sets and subsets?

EXAMPLE 4 Solving a problem using a Venn diagram

Bilyana recorded the possible sums that can occur when you roll two four-sided dice in an outcome table.

a) Display the following sets in one Venn diagram:
- rolls that produce a sum less than 5
- rolls that produce a sum greater than 5

b) Record the number of elements in each set.

c) Determine a formula for the number of ways that a sum less than or greater than 5 can occur. Verify your formula.

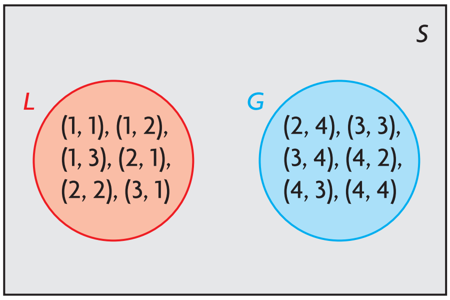
	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

Morgan's Solution

a) $S = \{\text{all possible sums}\}$
$L = \{\text{all sums less than 5}\}$
$G = \{\text{all sums greater than 5}\}$

I defined the universal set S and the subsets L and G.

I represented the relationship between the sums in a Venn diagram.



Since it is not possible for a sum to be less than 5 and greater than 5 at the same time, the subsets are disjoint. The events that describe the subsets are **mutually exclusive**.

mutually exclusive

Two or more events that cannot occur at the same time; for example, the Sun rising and the Sun setting are mutually exclusive events.

b) $n(S) = 16$
$n(L) = 6$
$n(G) = 6$

I counted the number of elements in each set using the outcome table.

c) The number of ways that these events can happen is the sum of the number of ways that each event can happen.

$$n(L \text{ or } G) = n(L) + n(G)$$
$$n(L \text{ or } G) = 6 + 6$$
$$n(L \text{ or } G) = 12$$

Events L and G are mutually exclusive. This is the sum of the number of elements in each set.

A sum that is greater than 5 or less than 5 can occur in 12 ways.

There are 4 ways that either sum cannot happen. Since the dice can fall in 16 ways, the sums that are greater than 5 or less than 5 can happen in 12 ways.

I verified my results using my table. There are 4 sums that are equal to 5 out of 16 possible sums.

Your Turn

Two six-sided dice are rolled.

a) Illustrate the following sets in one Venn diagram:
- rolls that produce a sum less than 6
- rolls that produce a sum greater than 6

b) Record the number of elements in each set.

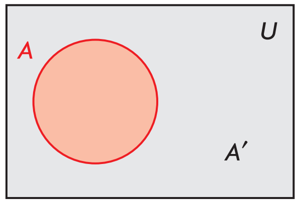
c) Determine a formula for the number of ways that a sum less than or greater than 6 can occur. Verify your formula.

In Summary

Key Ideas

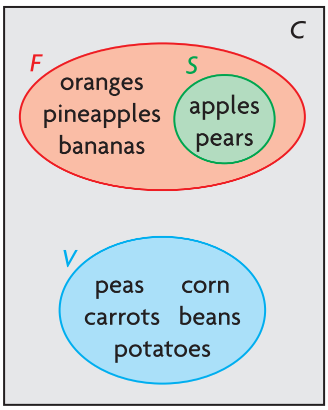
- You can represent a set of elements by:
  - listing the elements; for example, $A = \{1, 2, 3, 4, 5\}$
  - using words or a sentence; for example, $A = \{$all integers greater than 0 and less than 6$\}$
  - using set notation; for example, $A = \{x \mid 0 < x < 6, x \in I\}$
- You can show how sets and their subsets are related using Venn diagrams. Venn diagrams do not usually show the relative sizes of the sets.
- You can often separate a universal set into subsets, in more than one correct way.

Need to Know



- Sets are equal if they contain exactly the same elements, even if the elements are listed in different orders.
- You may not be able to count all the elements in a very large or infinite set, such as the set of real numbers.
- The sum of the number of elements in a set and its complement is equal to the number of elements in the universal set:
$$n(A) + n(A') = n(U)$$
- When two sets A and B are disjoint,
$$n(A \text{ or } B) = n(A) + n(B)$$

CHECK Your Understanding



1. Imelda drew the Venn diagram to the left.

a) Imelda described the sets as follows:
- $C = \{$produce$\}$
- $F = \{$fruit$\}$
- $S = \{$fruit you can eat without peeling$\}$
- $V = \{$vegetables$\}$

Do you agree with her descriptions?

b) Describe another way Imelda could define the sets in her diagram.

c) Why does it make sense that $S \subset F$ and $S \subset C$?

d) List the disjoint sets, if there are any.

e) Is F' equal to V? Explain.

f) Determine $n(V)$ using $n(F)$ and $n(C)$.

g) List the elements in S'.

2. a) Draw a Venn diagram to represent these sets:
   - the universal set U = {natural numbers from 1 to 40 inclusive}
   - E = {multiples of 8}
   - F = {multiples of 4}
   - S = {multiples of 17}

   b) List the disjoint subsets, if there are any.

   c) Is each statement true or false? Explain.
   - i) $E \subset F$
   - ii) $F \subset E$
   - iii) $E \subset E$
   - iv) F' = {odd numbers from 1 to 40}
   - v) In this example, the set of natural numbers from 41 to 50 is { }.

3. Nunavut (N) and the Northwest Territories (T) have the following fish species:
   - N = {walleye, northern pike, lake trout, Arctic char, Arctic grayling, lake whitefish}
   - T = {Arctic char, Arctic grayling, northern pike, lake trout, lake whitefish, inconnu, walleye}

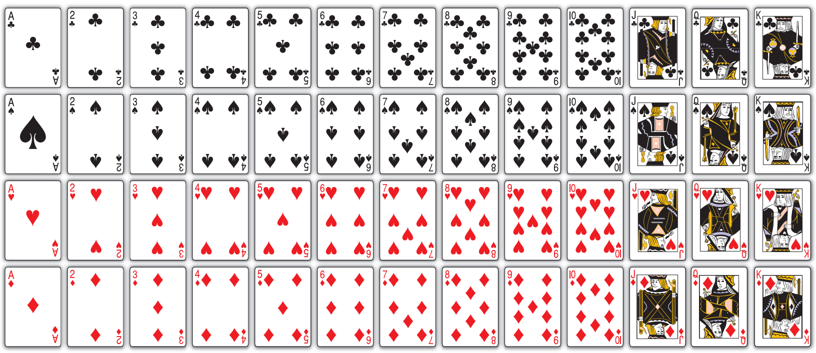
   a) Illustrate the sets of fish in these two territories using a Venn diagram.

   b) Explain what the following statement means: $N \subset T$, but $T \not\subset N$.

PRACTISING

4. For this question, the universal set, U, is a standard deck of 52 cards as shown.

   a) Represent the following sets and subsets using a Venn diagram:
   - B = {black cards}
   - R = {red cards}
   - S = {spades ♠}
   - H = {hearts ♥}
   - C = {clubs ♣}
   - D = {diamonds ♦}



   b) List all defined sets that are subsets of B.

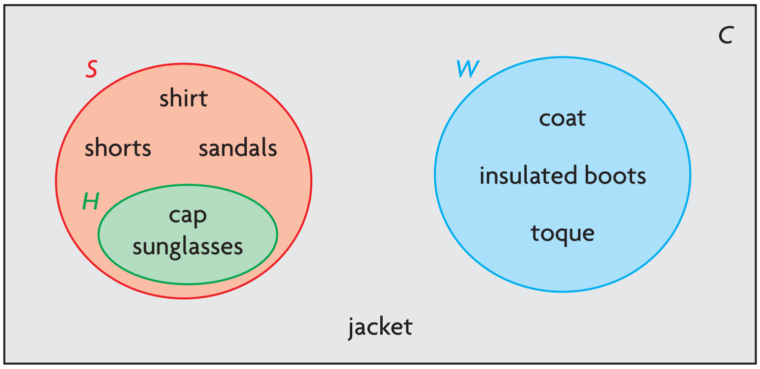
   c) List all defined sets that are subsets of R.

   d) Are sets S and C disjoint? Explain.

   e) Suppose you draw one card from the deck. Are the events drawing a heart and drawing a diamond mutually exclusive? Explain.

   f) Is the following statement correct?
   $n(S \text{ or } D) = n(S) + n(D)$
   Provide your reasoning. Determine the value of $n(S \text{ or } D)$.

5. Xavier drew this Venn diagram:



a) Describe what sets C, S, W, and H might represent.
b) Where would Xavier put running shoes?
c) Is S' equal to W? Explain.
d) List the disjoint sets, if there are any.
e) Categorize the items another way.

6. Consider the following information:
- the universal set $U = \{\text{natural numbers from 1 to 100 000}\}$
- $X \subset U$
- $n(X) = 12$

Determine $n(X')$, if possible. If it is not possible, explain why.

7. Consider the following information:
- the universal set $U = \{\text{all natural numbers from 1 to 10 000}\}$
- set X, which is a subset of U
- set Y, which is a subset of U
- $n(X) = 4500$

Determine $n(Y)$, if possible. If it is not possible, explain why.

8. Determine $n(U)$, the universal set, given $n(X) = 34$ and $n(X') = 42$.

9. Consider this universal set:
$A = \{$A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z$\}$
a) List the following subsets:
- $S = \{\text{letters drawn with straight lines only}\}$
- $C = \{\text{letters drawn with curves only}\}$

b) Is this statement true or false?
$C = S'$
Provide your reasoning.

10. Many forms of transportation can be used to travel around Alberta. These include walking, biking, driving, and skiing, as well as riding on buses, horses, airplanes, power boats, canoes, and hot-air balloons. Organize this information in a Venn diagram. Include other forms of transportation if you wish.

11. **a)** Organize the following sets of numbers in a Venn diagram:
    - $U = \{$integers from -10 to $10\}$
    - $A = \{$positive integers from 1 to 10 inclusive$\}$
    - $B = \{$negative integers from -10 to -1 inclusive$\}$

    b) List the disjoint subsets, if there are any.

    c) Is each statement true or false? Explain.
    - **i)** $A \subset B$
    - **ii)** $B \subset A$
    - **iii)** $A' = B$
    - **iv)** $n(A) = n(B)$
    - **v)** For set U, the set of integers from -20 to -15 is $\{ \}$.

12. Semiprime numbers are numbers that are the product of two prime numbers (1 is not considered to be a prime number). For example, 10 is a semiprime number because it is the product of 2 and 5.

    a) Use the set of natural numbers from 1 to 50, inclusive, as the universal set. Organize these numbers into the following sets:
    - $S = \{$semiprimes less than 50$\}$
    - $W = \{$other numbers$\}$

    b) Define one subset of S.

    c) Determine $n(W)$ without counting.

    d) Consider the set $A = \{$all semiprimes$\}$. Can you determine $n(A)$? Explain why or why not.

13. List items in your home that are related to entertainment or technology. Then organize these items into sets using a Venn diagram.

14. Cynthia claims that the $\subset$ sign for sets is similar to the $\leq$ sign for numbers. Explain whether you agree or disagree.

15. **a)** Indicate the multiples of 25 and 50, from -1000 to 1000, using set notation. List any subsets.

    b) Represent the sets and subsets in a Venn diagram.

16. Carol tosses a nickel, a dime, and a quarter. Each coin can turn up heads (H) or tails (T).

    a) List the elements of the universal set, U, for this situation.

    b) $E = \{$second coin turns up tails$\}$ List the elements of E.

    c) Determine $n(U)$ and $n(E)$.

    d) Is $E \subset U$?

    e) Describe E' in words. Determine $n(E')$ using $n(U)$ and $n(E)$. Confirm your answer by listing the elements of E'.

    f) Are E and E' disjoint sets? Explain.

17. a) Organize the following sets in a Venn diagram:

- the universal set $R = \{\text{real numbers}\}$
- $N = \{\text{natural numbers}\}$
- $W = \{\text{whole numbers}\}$
- $I = \{\text{integers}\}$
- $Q = \{\text{rational numbers}\}$
- $\overline{Q} = \{\text{irrational numbers}\}$

b) Identify the complement of each set.

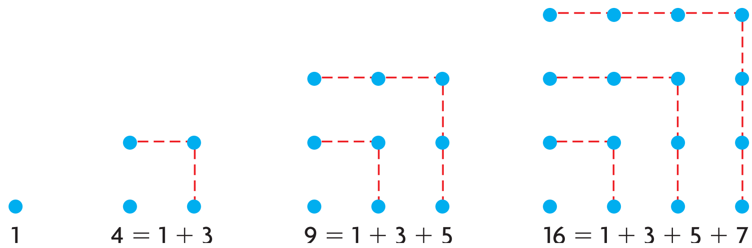
c) Identify any disjoint sets.

d) Are Q' and $\overline{Q}$ equal? Explain.

e) Of which sets is N a subset?

f) Joey drew a Venn diagram to show the sets in part a). In his diagram, the area of set Q was larger than the area of set $\overline{Q}$. Can you conclude that Q has more elements than $\overline{Q}$? Explain.

18. A square number can be represented as a square array.



Determine how many natural numbers from 1 to 300 inclusive are:

a) Even and square

b) Odd and square

c) Not square

Closing

19. Explain how to determine each of the following, and give an example.

a) Whether one set is a subset of another

b) Whether one set is a complement of another

Extending

20. Do you agree or disagree with the following explanation? Explain. Consider the set $U = \{\text{natural numbers from 20 to 30}\}$. One empty subset of U is the set of natural numbers from 1 to 19. Another empty subset is the natural numbers greater than 21. Therefore, U has two different empty subsets, not one.